Applied Statistics for Life Sciences

Dmitri D. Pervouchine

Centre de Regulació Genòmica

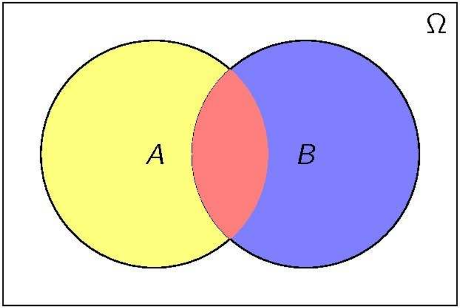
Module 2: Probability Review

Contents

Probability

Probability is a *normalized measure*

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- $0 \leq P(A) \leq 1$
- $P(\Omega) = 1$



Problem 1.1

Given that $P(A) = 0.6$ *and* $P(B) = 0.7$, *which of the following* **cannot** *be true? [Recall that* $\cup = OR$; $\cap = AND$*]*

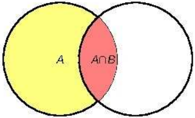
1. $P(A \cap B) = 0.5$
2. $P(A \cup B) = 0.9$
3. $P(A \cap B) = 0.2$
4. $P(A \cup B) = 0.4$
5. $P(A \cap B) = 0.7$

Solution

Cannot be true:

3. $P(A \cap B) = P(A) + P(B) - P(A \cup B) \geq P(A) + P(B) - 1 = 0.3$
4. $P(A \cup B) \geq \max\{P(A), P(B)\} = 0.7$
5. $P(A \cap B) \leq \min\{P(A), P(B)\} = 0.6$

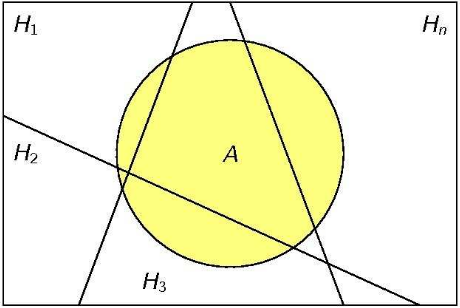
Conditional Probability



- A and B are called independent if $\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B)$
- $\mathrm{P}(A|B) = \frac{\mathrm{P}(A \cap B)}{\mathrm{P}(B)}$
- $\mathrm{P}(A|B) =$ the fraction of A in B
- A and B are independent iff $\mathrm{P}(A|B) = P(A)$

Law of Total Probability

- $P(A) = P(A|H_1)\,P(H_1) + P(A|H_2)\,P(H_2) + \ldots + P(A|H_n)\,P(H_n)$
- In particular, $P(A) = P(A|B)\,P(B) + P(A|\overline{B})\,P(\overline{B})$



Bayes Formula

- $\mathrm{P}(A|B) = \frac{\mathrm{P}(B|A)\,\mathrm{P}(A)}{\mathrm{P}(B)}$

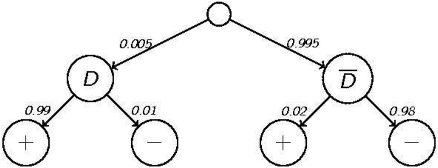
- $\mathrm{P}(B|A) =$ prior probability

- $\mathrm{P}(A|B) =$ posterior probability

Problem 1.2

Suppose a certain drug test is 99% sensitive and 98% specific, that is, the test will correctly identify a drug user as testing positive 99% of the time and will correctly identify a non-user as testing negative 98% of the time. Let's assume a corporation decides to test its employees for opium use, and 0.5% of the employees use the drug. What is the probability that, given a positive drug test, an employee is actually a drug user?

Solution



$$P(D|+) = 0.005 \cdot 0.99/(0.005 \cdot 0.99 + 0.995 \cdot 0.02) = 0.1991$$

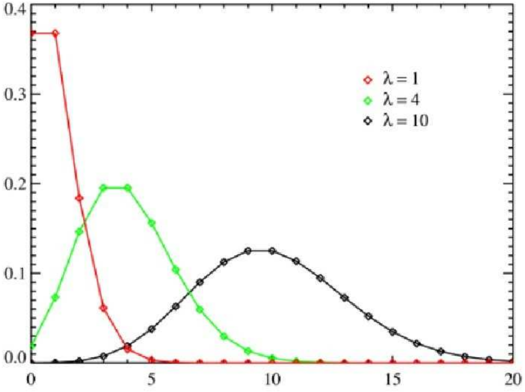
Distributions

- Discrete (Uniform, Binomial, Poisson, Geometric, Hypergeometric, Negative Binomial, . . .)

- Continuous (Uniform, Normal, Exponential, Gamma, Chi-square, Student, Fisher, Dirichlet, . . .)

Discrete Distributions

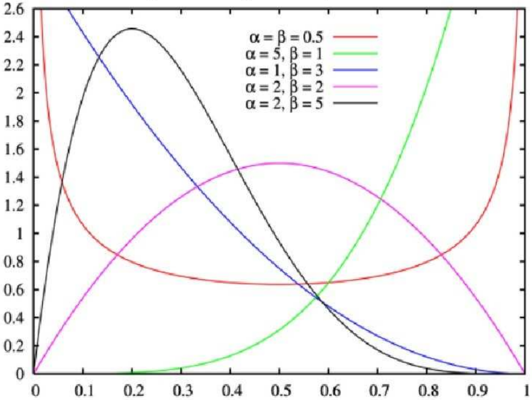
In a discrete distribution, probability is assigned to isolated values



Poisson distribution

Continuous Distributions

In a continuous distribution, probability density is smirred over a range of values



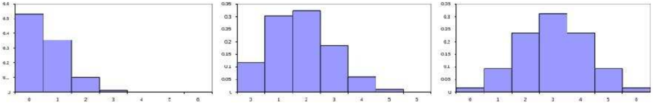
Beta distribution

Convention on the Notation for Random Numbers

- Random numbers will be denoted by capitals $X, Y, Z, \dots$
- Regular (non-random numbers) will be denoted by small $x, y, z, \dots$
- The expression "$X = 1$" makes no sense
- The expression $\mathrm{P}(X = 1) = 0.45$ does make sense

Binomial Distribution

- Binomial random number = the number of successes in n independent trials; p is the probability of success in one trial
- $\mathrm{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k}$
- $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, where $k! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot k$



p=0.1 p=0.3 p=0.5

Problem 2.1

The probability that a certain machine will produce a defective item is 0.20. If a random sample of 6 items is taken from the output of this machine, what is the probability that there will be exactly 2 defectives in the sample?

Solution

$$P(1 \cap 2) = 0.2 \cdot 0.2 \cdot 0.8 \cdot 0.8 \cdot 0.8 \cdot 0.8 = 0.2^2 \cdot 0.8^4$$

$$P(2 \cap 5) = 0.8 \cdot 0.2 \cdot 0.8 \cdot 0.8 \cdot 0.2 \cdot 0.8 = 0.2^2 \cdot 0.8^4$$

$$P = \binom{6}{2} 0.2^2 \cdot 0.8^4 = \frac{6 \cdot 5}{2 \cdot 1} 0.2^2 \cdot 0.8^4 \simeq 0.2458$$

Problem 2.2

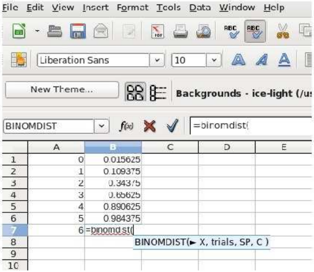
There are 10 patients on the Neo-Natal Ward of a local hospital who are monitored by 2 staff members. If the probability (at any one time) of a patient requiring emergency attention by a staff member is 0.3, assuming the patients to be behave independently, what is the probability at any one time that there will not be sufficient staff to attend all emergencies?

Solution

- $X = \textit{the number of emergencies}$
- $X \sim Bi(n = 10, p = 0.3)$
- $\mathrm{P}(X > 2) = 1 - \mathrm{P}(X \leq 2) = 1 - \mathrm{P}(X = 0) - \mathrm{P}(X = 1) - \mathrm{P}(X = 2) =$
  $1 - \binom{10}{0} 0.3^0 \cdot 0.7^{10} - \binom{10}{1} 0.3^1 \cdot 0.7^9 - \binom{10}{2} 0.3^2 \cdot 0.7^8 = 0.6172$

Cumulative Probability, C.D.F.

- Cumulative Density Function $F(x) = \mathrm{P}(X \leq x)$
- For Binomial Distribution: $F(x; n; p) = \sum_{k=0}^{k \leq x} \mathrm{P}(X = k)$

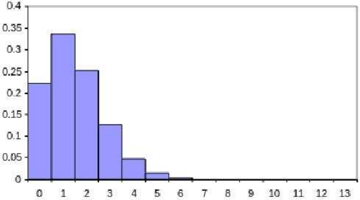


- In R-statistics $p = \texttt{pbinom(x, n, p)}$

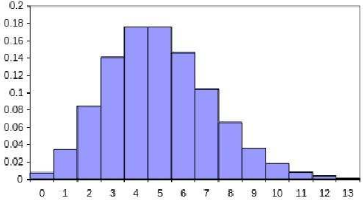
Poisson Distribution

Poisson random number = the number of rare events per unit of time or space; λ is the intensity parameter

$$P(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}$$



λ=1.5



λ=5

Problem 2.3

The marketing manager of a company has noted that she usually receives 10 complaint calls during a week (consisting of five working days), and that the calls occur at random. Find the probability that she gets five such calls in one day.

Solution

- $X =$ *the number of complaint calls per day*
- $X \sim Poisson(\lambda = 2)$
- $\mathrm{P}(X = 5) = \frac{2^5}{5!}e^{-2} = 0.0361$

Problem 2.4

The rate at which a particular defect occurs in lengths of plastic film being produced by a stable manufacturing process is 4.2 defects per 75 meter length. A random sample of the film is selected and it was found that the length of the film in the sample was 25 meters. What is the probability that there will be at most 2 defects found in the sample?

Solution

- $X =$ *the number of defects per 25 meters*
- $X \sim Poisson(\lambda = 1.4)$
- $\mathrm{P}(X \leq 2) = e^{-1.4}\left(\frac{1.4^0}{0!} + \frac{1.4^1}{1!} + \frac{1.4^2}{2!}\right) = 0.8335$

Poisson Approximation to Binomial

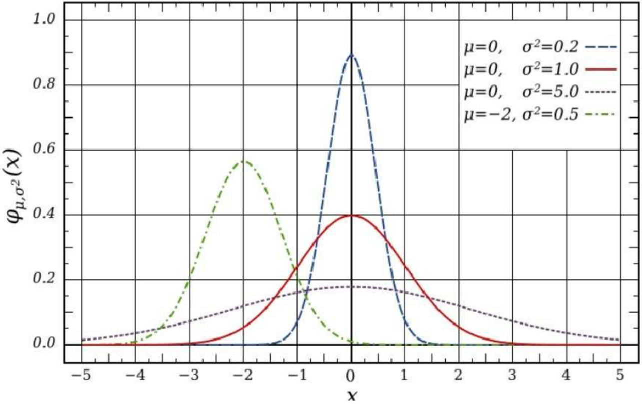
- 4.2 defects per 75 meter length
- $n = 25$ meters with $p = \frac{1.4}{25}$ probability of defect in each meter
- $n = 2500$ cm with $p = \frac{1.4}{2500}$ probability of defect in each centimeter
- $n = 25000$ mm with $p = \frac{1.4}{25000}$ probability of defect in each millimeter
- $$n \to \infty, p \to 0, np = \lambda = const$$

  $$n >> k$$

$$\mathrm{P}(X = k) = \binom{n}{k} p^k (1-p)^{n-k} = \frac{n(n-1)(n-2)\ldots(n-k+1)}{k(k-1)(k-2)\ldots1} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^{n-k} \simeq \frac{n^k}{k(k-1)(k-2)\ldots1} \left(\frac{\lambda}{n}\right)^k \left(1 - \frac{\lambda}{n}\right)^n = \frac{\lambda^k}{k!} \left(1 - \frac{\lambda}{n}\right)^n \simeq \frac{\lambda^k}{k!} e^{-\lambda}$$

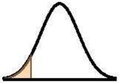
Normal Distribution

Normal random number = contribution of many independent random factors; $\mu =$ measure of center, $\sigma =$ measure of spread.



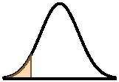
Cumulative Probability Tables

Standard normal distribution: $\mu = 0$ and $\sigma = 1$.



z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
-3.4	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0003	.0002
-3.3	.0005	.0005	.0005	.0004	.0004	.0004	.0004	.0004	.0004	.0003
-3.2	.0007	.0007	.0006	.0006	.0006	.0006	.0006	.0005	.0005	.0005
-3.1	.0010	.0009	.0009	.0009	.0008	.0008	.0008	.0008	.0007	.0007
-3.0	.0013	.0013	.0013	.0012	.0012	.0011	.0011	.0011	.0010	.0010
-2.9	.0019	.0018	.0018	.0017	.0016	.0016	.0015	.0015	.0014	.0014
-2.8	.0026	.0025	.0024	.0023	.0023	.0022	.0021	.0021	.0020	.0019
-2.7	.0035	.0034	.0033	.0032	.0031	.0030	.0029	.0028	.0027	.0026
-2.6	.0047	.0045	.0044	.0043	.0041	.0040	.0039	.0038	.0037	.0036
-2.5	.0062	.0060	.0059	.0057	.0055	.0054	.0052	.0051	.0049	.0048
-2.4	.0082	.0080	.0078	.0075	.0073	.0071	.0069	.0068	.0066	.0064
-2.3	.0107	.0104	.0102	.0099	.0096	.0094	.0091	.0089	.0087	.0084
-2.2	.0139	.0136	.0132	.0129	.0125	.0122	.0119	.0116	.0113	.0110
-2.1	.0179	.0174	.0170	.0166	.0162	.0158	.0154	.0150	.0146	.0143
-2.0	.0228	.0222	.0217	.0212	.0207	.0202	.0197	.0192	.0188	.0183
-1.9	.0287	.0281	.0274	.0268	.0262	.0256	.0250	.0244	.0239	.0233
-1.8	.0359	.0351	.0344	.0336	.0329	.0322	.0314	.0307	.0301	.0294
-1.7	.0446	.0436	.0427	.0418	.0409	.0401	.0392	.0384	.0375	.0367
-1.6	.0548	.0537	.0526	.0516	.0505	.0495	.0485	.0475	.0465	.0455
-1.5	.0668	.0655	.0643	.0630	.0618	.0606	.0594	.0582	.0571	.0559
-1.4	.0808	.0793	.0778	.0764	.0749	.0735	.0721	.0708	.0694	.0681
-1.3	.0968	.0951	.0934	.0918	.0901	.0885	.0869	.0853	.0838	.0823
-1.2	.1151	.1131	.1112	.1093	.1075	.1056	.1038	.1020	.1003	.0985
-1.1	.1357	.1335	.1314	.1292	.1271	.1251	.1230	.1210	.1190	.1170
-1.0	.1587	.1562	.1539	.1515	.1492	.1469	.1446	.1423	.1401	.1379
-0.9	.1841	.1814	.1788	.1762	.1736	.1711	.1685	.1660	.1635	.1611
-0.8	.2119	.2090	.2061	.2033	.2005	.1977	.1949	.1922	.1894	.1867
-0.7	.2420	.2389	.2358	.2327	.2296	.2266	.2236	.2206	.2177	.2148
-0.6	.2743	.2709	.2676	.2643	.2611	.2578	.2546	.2514	.2483	.2451
-0.5	.3085	.3050	.3015	.2981	.2946	.2912	.2877	.2843	.2810	.2776
-0.4	.3446	.3409	.3372	.3336	.3300	.3264	.3228	.3192	.3156	.3121
-0.3	.3821	.3783	.3745	.3707	.3669	.3632	.3594	.3557	.3520	.3483
-0.2	.4207	.4168	.4129	.4090	.4052	.4013	.3974	.3936	.3897	.3859
-0.1	.4602	.4562	.4522	.4483	.4443	.4404	.4364	.4325	.4286	.4247
-0.0	.5000	.4960	.4920	.4880	.4840	.4801	.4761	.4721	.4681	.4641

Cumulative Probability Tables



z	.00	.01	.02	.03	.04
-3.4	.0003	.0003	.0003	.0003	.0003
-3.3	.0005	.0005	.0005	.0004	.0004
-3.2	.0007	.0007	.0006	.0006	.0006
-3.1	.0010	.0009	.0009	.0009	.0008
-3.0	.0013	.0013	.0013	.0012	.0012
-2.9	.0019	.0018	.0018	.0017	.0016
-2.8	.0026	.0025	.0024	.0023	.0023
-2.7	.0035	.0034	.0033	.0032	.0031
-2.6	.0047	.0045	.0044	.0043	.0041
-2.5	.0062	.0060	.0059	.0057	.0055
-2.4	.0082	.0080	.0078	.0075	.0073
-2.3	.0107	.0104	.0102	.0099	.0096
-2.2	.0139	.0136	.0132	.0129	.0125
-2.1	.0179	.0174	.0170	.0166	.0162
-2.0	.0228	.0222	.0217	.0212	.0207

$$P(Z < -2.51) = 0.0060$$

Other Normal Distributions

- $Z \sim \mathcal{N}(0,1)$
  - Mean $= 0$
  - Standard deviation $= 1$

- $X \sim \mathcal{N}(\mu, \sigma)$
- Note that some authors denote $\mathcal{N}(\mu, \sigma^2)$
  - Mean $= \mu$
  - Standard deviation $= \sigma$

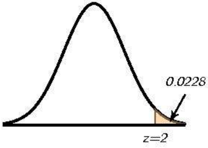
- $Z = (X - \mu)/\sigma$

Problem 2.5

The diameters of steel disks produced in a plant are normally distributed with a mean of 2.5 cm and standard deviation of 0.02 cm. What is the probability that a disk picked at random has a diameter greater than 2.54 cm?

Solution

- X = *diameters of a randomly picked steel disk*
- $X \sim \mathcal{N}(\mu = 2.5, \sigma = 0.02)$
- $P(X > 2.54) = P(X - \mu > 2.54 - 2.5) = P(\frac{X-\mu}{\sigma} > \frac{2.54-2.5}{0.02}) = P(Z > 2)$



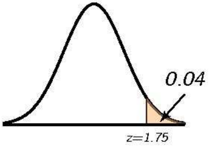
$P(Z > 2) = P(Z < -2) = 0.0228$

Problem 2.6

The height of an adult male is known to be normally distributed with a mean of 69 inches and a standard deviation of 2.5 inches. What is the height of the doorway such that 96 percent of the adult males can pass through it without having to bend?

Solution

- X = *height of a randomly picked adult male* $\sim \mathcal{N}(\mu = 69, \sigma = 2.5)$
- $h = ?$ *such that* $\mathrm{P}(X > h) = 0.04$
- $\mathrm{P}(X > h) = \mathrm{P}(X - \mu > h - 69) = \mathrm{P}(\frac{X-\mu}{\sigma} > \frac{h-69}{2.5}) = \mathrm{P}(Z > \frac{h-69}{2.5}) = 0.04$



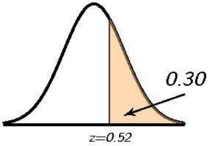
$\frac{h-69}{2.5} = 1.75, h = 73.38 \text{ in}$

Problem 2.7

The longevity of people living in a certain locality has a standard deviation of 14 years. What is the mean longevity if 30% of the people live longer than 75 years? Assume a normal distribution for life spans.

Solution

- X = *the longevity of a randomly chosen person* $\sim \mathcal{N}(\mu = ?, \sigma = 14)$
- $\mu = ?$ *given that* $P(X > 75) = 0.30$
- $P(X > 75) = P(Z > \frac{75-\mu}{14}) = 0.30$



$\frac{75-\mu}{14} = 0.52, \mu = 67.72$

Normal Approximation to Binomial Distribution

- $X = Binom(n, p)$
- $n =$ number of trials
- $p =$ probability of a success in one trial

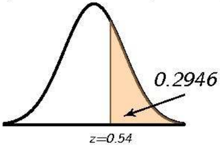
- $X = \mathcal{N}(\mu, \sigma)$
- $\mu = np$
- $\sigma = \sqrt{np(1-p)}$
- Approximation valid when $n > 40$, $np > 5$, and $n(1-p) > 5$

Problem 2.8

The unemployment rate in a certain city is 8.5%. A random sample of 100 people from the labor force is drawn. Find the approximate probability that the sample contains at least ten unemployed people.

Solution

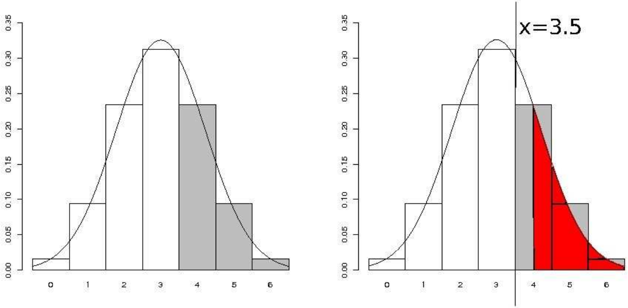
- X = *the number of unemployed people* $\sim Bi(n = 100, p = 0.085)$
- $X \sim \mathcal{N}(\mu = np = 8.5, \sigma = \sqrt{np(1-p)} = 2.79)$
- $\mathrm{P}(X \geq 10) = \mathrm{P}(Z > \frac{10-8.5}{2.79}) = \mathrm{P}(Z > 0.54)$



$\mathrm{P}(Z > 0.54) = 0.2946$

Continuity Correction

The concept of continuity correction applies when a discrete distribution is approximated by a continuous distribution.



- $\Pr(X \geq 4)$ by binomial distribution is gray
- $\Pr(X \geq 4)$ by normal distribution is red
- Need to step 0.5 units left

Continuity Correction Rule

X = discrete variable

Y = continuous variable

Write integer inequality in two forms

$$P(X \le k) = P(X < k + 1)$$

and take the average of the two borders, i.e.

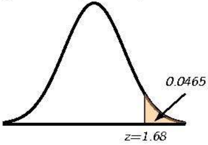
$$P(X \le k) = P(X < k + 1) \simeq P(Y < k + \frac{1}{2})$$

Problem 2.9

Companies are interested in the demographics of those who listen to the radio programs they sponsor. A radio station has determined that only 20% of listeners phoning in to a morning talk program are male. During a particular week, 200 calls are received by this program. What is the approximate probability that at least 50 of the callers are male?

Solution

- $X \sim Bi(n = 200, p = 0.20) \sim \mathcal{N}(\mu = np = 40, \sigma = \sqrt{np(1-p)} = 5.66)$
- $\mathrm{P}(X \geq 50) = \mathrm{P}(X > 49) = \mathrm{P}(Z > \frac{49.5-40}{5.66}) = \mathrm{P}(Z > 1.68)$



$\mathrm{P}(Z > 1.68) = 0.0465$

- *Without continuity correction* $P = \mathrm{P}(Z > \frac{50-40}{5.66}) = \mathrm{P}(Z > 1.77) = 0.0384$

  Exact probability $P = 1 - \texttt{pbinom}(49, 200, 0.2) = 0.0494$

Expected Value and Variance

Random number = values with assigned probabilities

x	x_1	x_2	...	...	x_n
p	p_1	p_2	...	...	p_n

- Expected value $\mathrm{E}(X) = \sum_{i=1}^{n} x_i p_i$ = not a random number
- Expected value is a measure of center
- $\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$
- Variance is a measure of spread
- Standard deviation $\sigma = \sqrt{\mathrm{Var}(X)}$

Good Properties of Expected Value and Variance

- $\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)$
- $\mathrm{E}(cX) = c\,\mathrm{E}(X)$
- $\mathrm{E}(c) = c$
- If X and Y are independent then $\mathrm{E}(XY) = \mathrm{E}(X)\,\mathrm{E}(Y)$

- $\mathrm{Var}(X) = \mathrm{E}(X^2) - E^2(X)$
- $\mathrm{Var}(cX) = c^2\,\mathrm{Var}(X)$
- If X and Y are independent then $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y)$ and $\sigma(X+Y) = \sqrt{\sigma^2(X) + \sigma^2(Y)}$
- $\mathrm{Var}(X+Y) = \mathrm{Var}(X) + \mathrm{Var}(Y) + 2\,\mathrm{Cov}(X,Y)$
- Note that standard deviation follows Pythagorean rule $c = \sqrt{a^2 + b^2}$

Problem 2.10

The Attila Barbell Company makes bars for weight lifting. The weights of the bars are independent and are normally distributed with a mean of 720 ounces (45 pounds) and a standard deviation of 4 ounces. The bars are shipped 10 in a box to the retailers. The weights of the empty boxes are normally distributed with a mean of 320 ounces and a standard deviation of 8 ounces. The weights of the boxes filled with 10 bars are expected to be normally distributed with a mean of 7,520 ounces. What is the standard deviation?

Solution

- $X_i = \textit{the weight of the i-th bar} \sim \mathcal{N}(\mu = 720, \sigma = 4), i = 1 \dots 10$
- $Y = \textit{the weight of the box} \sim \mathcal{N}(\mu = 320, \sigma = 8)$
- $W = \textit{the weight of the package} = X_1 + \cdots + X_{10} + Y$
- $\mathrm{E}(W) = \mathrm{E}(X_1) + \cdots + \mathrm{E}(X_{10}) + \mathrm{E}(Y) = 10 * 720 + 320 = 7,520$
- $\mathrm{Var}(W) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_{10}) + Var(Y) = 10 * 4^2 + 8^2 = 224$
- $\sigma(W) = \sqrt{224} \simeq 15\ oz$
- $\textit{Note that}\ \mathrm{Var}(10X + Y) = 10^2 * 4^2 + 8^2 = 1664 >> 224$

Sampling distribution

- Sample $X_1, X_2, \dots, X_n$
- X_i are random numbers from certain population

Population = heights of adult males

Assume that X_i

- are from the same distribution
- are independent

X_1	X_2	X_3
176	181	190
181	190	176
190	176	181
164	176	188
190	190	190
...	...	...
...	...	...
...	...	...

Sample Mean

Sample mean is defined as $\overline{X} = \frac{X_1+X_2+...+X_n}{n}$

Assume that

- X_i are from the same distribution
  - $E(X_i) = \mu$
  - $Var(X_i) = \sigma^2$
- X_i and X_j are independent for $i \neq j$

$$E(\overline{X}) = E\left(\frac{X_1+X_2+...+X_n}{n}\right) = \frac{1}{n}E(X_1 + X_2 + \ldots + X_n) = \frac{1}{n}(\mu + \mu + \ldots + \mu) = \frac{n\mu}{n} = \mu$$

The expected value of sample mean is the population mean

Law of Large Numbers

$\overline{X} = \frac{X_1+X_2+\ldots+X_n}{n}$

$\text{Var}(\overline{X}) = \text{Var}\left(\frac{X_1+X_2+\ldots+X_n}{n}\right) = \frac{1}{n^2}\text{Var}(X_1 + X_2 + \ldots + X_n) = \frac{1}{n^2}(\sigma^2 + \sigma^2 + \ldots + \sigma^2) = \frac{n\sigma^2}{n^2} = \sigma^2/n$

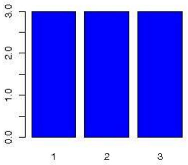
$\text{Var}(\overline{X}) = \frac{\text{Var}(X)}{n}$ and $\sigma(\overline{X}) = \frac{\sigma(X)}{\sqrt{n}}$

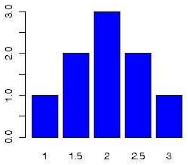
The larger the sample size, the less spread is the distribution of $\overline{X}$.

A Specific Example

X_1	X_2	$\overline{X}$
1	1	1
1	2	1.5
1	3	2
2	1	1.5
2	2	2
2	3	2.5
3	1	2
3	2	2.5
3	3	3

Population $= \{1, 2, 3\}$, sample size $n = 2$

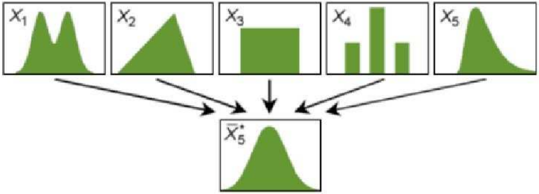




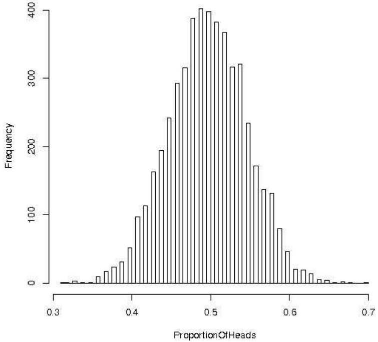
Central Limit Theorem

Theorem 1

The sum of a sufficiently large number of identically distributed independent random variables is approximately normally distributed regardless of the population distribution.



Normal Approximation to Binomial Distribution



X = number of successes in n trials

$$X = X_1 + X_2 + \ldots + X_n$$

$$X_i = \begin{cases} 0, & \text{if no success} \\ 1, & \text{if success} \end{cases}$$

$$\mathrm{E}(X_i) = p$$

$$\mathrm{Var}(X_i) = p - p^2 = p(1-p)$$

$$\mathrm{E}(X) = np$$

$$\sigma(X) = \sqrt{np(1-p)}$$

Problem 3.1

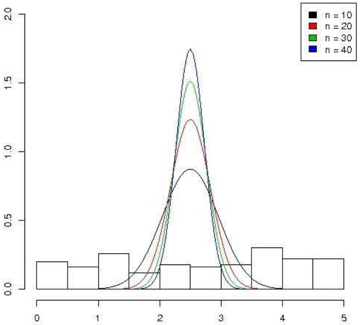
There are two games involving flipping a coin. In the first game you win a prize if you can throw between 45% and 55% of heads. In the second game you win if you can throw more than 80% heads. For each game would you rather flip the coin 30 times or 300 times?

Solution

Throw between 45% and 55% of heads = smaller variance = larger sample

Throw more than 80% heads = larger variance = smaller sample

Sampling distribution



$\overline{X}$ is approximately normal when $n > 40$

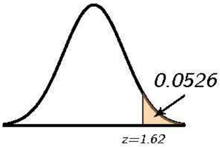
$\overline{X}$ is approximately normal *regardless* of the distribution of X

Problem 3.2

The average outstanding bill for delinquent customer accounts for a national department store chain is $187.50 with a standard deviation of $54.50. In a simple random sample of 50 delinquent accounts, what is the probability that the mean outstanding bill is over $200?

Solution

- X = *random outstanding bill for a delinquent customer*
- $\mathrm{E}(X) = \mu_X = 187.50, \quad \sigma_X = 54.50$
- $\overline{X} \sim \mathcal{N}(\mu = 187.50, \sigma = \frac{54.50}{\sqrt{50}})$
- $\mathrm{P}(\overline{X} > 200) = \mathrm{P}(Z > \frac{200-187.50}{54.50/\sqrt{50}}) = \mathrm{P}(Z > 1.62)$



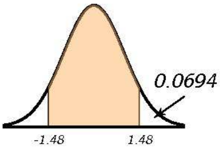
$\mathrm{P}(Z > 1.62) = 0.0526$

Problem 3.3

The average number of daily emergency room admissions at a hospital is 85 with standard deviation of 37. In a simple random sample of 30 days, what is the probability that the mean number of daily emergency admissions is between 75 and 95?

Solution

- X = *the number of daily emergency room admissions*
- $\mu_X = 85, \quad \sigma_X = 37$
- $\overline{X} \sim \mathcal{N}(\mu = 85, \sigma = \frac{37}{\sqrt{30}})$
- $\mathrm{P}(75 < \overline{X} < 95) = \mathrm{P}(\frac{75-85}{37/\sqrt{30}} < Z < \frac{95-85}{37/\sqrt{30}}) = \mathrm{P}(-1.48 < Z < 1.48)$



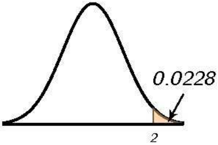
$$\mathrm{P}(-1.48 < Z < 1.48) = 1 - 2 * 0.0694 = 0.8612$$

Problem 3.4

A summer resort rents rowboats to customers but does not allow more than four people to a boat. Each boat is designed to hold no more than 800 pounds. Suppose the distribution of adult males who rent boats, including their clothes and gear, is normal with a mean of 190 pounds and standard deviation of 10 pounds. If the weights of individual passengers are independent, what is the probability that a group of four adult male passengers will exceed the acceptable weight limit of 800 pounds?

Solution

- $X = \textit{weight of a passenger} \sim \mathcal{N}(\mu = 190, \sigma = 10)$
- $\mathrm{P}(X_1 + X_2 + X_3 + X_4 > 800) = \mathrm{P}(\overline{X} > 200) = ?$



- $\mathrm{P}(\overline{X} > 200) = \mathrm{P}(Z > \frac{200-190}{10/\sqrt{4}}) = \mathrm{P}(Z > 2) = 0.0228$

Population normality

- If n is large then $\overline{X}$ is approximately normal regardless of the population distribution
- A non-trivial linear combination of independent normal distributions is normal
- If n is small AND population is normal then $\overline{X}$ is also normal
- For large n, population normality is **not** required
- For small n, population normality **is** required

Summary

- Probability is a non-negative measure normalized to 1
- Independent events are such that $\mathrm{P}(A \cap B) = \mathrm{P}(A) \cdot \mathrm{P}(B)$
- Discrete distributions take only isolated values, while continuous distributions also take all intermediate values
- Continuity correction is needed when approximating a discrete distribution by a continuous distribution
- Expected value is a linear operation, i.e. $\mathrm{E}(X+Y) = \mathrm{E}(X) + \mathrm{E}(Y)$
- Standard deviation is not, namely if X and Y are independent then $\sigma(X+Y) = \sqrt{\sigma^2(X) + \sigma^2(Y)}$ (Pythagorean theorem)
- Sampling distribution if the distribution of sample means; its mean is the same as in the population, but σ is $\sqrt{n}$ times smaller (The Law of Large Numbers)
- Sample mean is approximately normally distributed when n is large (Central Limit Theorem)